Topics from harmonic analysis related to generalized Poincaré-Sobolev inequalities: Lecture IV

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A digression: concavity and BMO

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(Motivated by work of A. A. Logunov, L. Slavin, D. M. Stolyarov, V. Vasyunin, and P. B. Zatitskiy,)

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Thm There exists a positive constant C such that

$$\sup_{t>0} t^p |\{y \in \mathbb{R}^n : Mf(y) > t\}| \leq C \sup_{t>0} t^p \Big|\{y \in \mathbb{R}^n : M^\# f(y) > t\}\Big|$$

for all functions f such that the left side is finite.

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The typical following **good–λ inequality** is: there is a universal constant q such that for any $t > 0$, and $0 < \epsilon < \epsilon_0$,

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The typical following **good–λ inequality** is: there is a universal constant q such that for any $t > 0$, and $0 < \epsilon < \epsilon_0$,

$$\Big|\{y \in \mathbb{R}^n : Mf(y) > qt, M^{\#} f(y) \leq t\epsilon\}\Big| \leq C\,\epsilon |\{y \in \mathbb{R}^n : Mf(y) > t\}|$$

Good-λ method: Proof I

Now, for each $t > 0$, we let $\Omega_t = \{x \in \mathbb{R}^n : Mf(x) > t\}$ We can consider the Calderón–Zygmund covering lemma of f in $\mathbb{R}^n$ for these values of t. This yields a collection of dyadic cubes $\{Q_i\}$, maximal with respect to inclusion, satisfying $\Omega_t = \cup_i Q_i$ and

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On the other hand for arbitrary x,

$$|f(x)| \leq |f(x) - f_{Q_i}| + |f_{Q_i}| \leq |f(x) - f_{Q_i}| + \frac{1}{|Q_i|}\int_{Q_i} |f| \leq |f(x) - f_{Q_i}| + 2^n\, t$$

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For I we use that M is of weak type $(1,1)$ to control the unweighted part:

$$\left|E_{Q_i}\right| \leq \frac{1}{(q - 2^n)t}\frac{1}{|Q_i|}\int_{Q_i} |f - f_{Q_i}|\,|Q_i| \leq \frac{\epsilon}{(q - 2^n)}|Q_i|.$$

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Combining we have that for $q > 2^n$ (say $q = 2^n + 1$) and $t > 0$,

$$(qt)^p\,|\Omega_{qt}| \leq \epsilon\, q^p\, t^p\, |\Omega_t| + (tq)^p |\{x \in \mathbb{R}^n : M^{\#}f(x) > \epsilon t\}|$$

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Good-λ method: Proof III

Then
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END FOURTH LECTURE